Infinite resistive lattices

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The resistance between two arbitrary nodes in an infinite square lattice of identical resistors is calculated. The method is generalized to infinite triangular and hexagonal lattices in two dimensions, and also to infinite cubic and hypercubic lattices in three and more dimensions.

I. INTRODUCTION

The resistance between two arbitrary nodes in an infinite lattice made of identical resistors is calculated. The method was introduced by Venezian,[1] and works by superposing the potentials and currents corresponding to two configurations in each of which only one finite node carries a current from outside the lattice. The currents in such a uniterminal circuit enjoy the full symmetry of the lattice, whereas the currents of the diterminal circuit that is actually of interest have lower symmetry. Use of a similar higher symmetry was analogously made in a recent treatment of the resistance between the vertices of regular polyhedra constructed from identical resistors.[2]

Despite the emphasis on the high symmetry of a uniterminal lattice, the method proceeds, oddly enough, through the imposition of an ansatz that does not possess the full symmetry in a manifest manner. By using complex Fourier transforms we illuminate the method of Venezian and generalize it from the original square lattice in two dimensions to cubic and hypercubic lattices in higher dimensions, as well as to triangular and hexagonal lattices in two dimensions.

The final result for the resistance between two nodes is expressed as an integral transform, and it does not seem possible to express this as a known higher transcendental function, like a generalized hypergeometric function, for example. Nevertheless, for specific choices of the lattice, and of the nodes, it turns out to be possible to evaluate the integrals algebraically, in terms of π and of the square roots of integers. It was within the capacity of Mathematica 3.0 to evaluate sample results for two-dimensional lattices, yielding exact answers in place of the numerical approximations given in Venezian's paper.

Special cases of the results obtained in this paper have been published before. The resistance between adjacent mesh points in the square lattice was derived by Aitchison,[3] while Bartis[4] treated adjacent mesh points in more general two-dimensional lattices. Purcell[5] gives the resistance between *diagonally opposed* mesh points of the square lattice without proof, while a proof of this result is provided by Lavatelli,[6] who emphasizes how the resistive net can be thought of as a discrete approximation to a continuous resistive medium. Trier[7] obtains a *double* integral representation for the resistances in the case of the square lattice, together with a table of exact results; this work was further evaluated by Cameron,[8] who considered carefully the conditions for the existence and uniqueness of Trier's solution. In the present work we tacitly obtain uniqueness by requiring that the currents at infinity vanish, and this condition accords with one of Cameron's criteria. Finally, the work of Zemanian[9] should be noted: he also considers the uniqueness problem, as well as the double integral representation for the square lattice, and analogous expressions for uniform and nonuniform lattices in more than two dimensions.

II. TWO-DIMENSIONAL SQUARE LATTICE

Consider an infinite square lattice formed from equal resistors, and let (n,p) be the node that is n lattice units in the horizontal, and p units in the vertical direction from an arbitrarily chosen origin. It is supposed that a current of I_{np} A can enter the (n,p) node from a source outside the lattice (see Fig. 1).

If the potential at this node is denoted by V_{np} V, we have, by a combination of Ohm's and Kirchhoff's laws,

$$\begin{aligned} I_{np} &= (V_{np}-V_{n+1,p})+(V_{np}-V_{n-1,p})+(V_{np}-V_{n,p+1}) \\ &\quad +(V_{np}-V_{n,p-1}) \\ &= 4V_{np}-V_{n+1,p}-V_{n-1,p}-V_{n,p+1}-V_{n,p-1}, \end{aligned}$$

where, without essential loss of generality, we have taken each resistor to be of 1 Ω. In general, a current may be injected at every node from the outside.

We shall seek an integral representation for the potential at the node (n,p) in the form

$$V_{np} = \int_0^{2\pi} d\beta F(\beta) v_{np}(\beta),$$

with

$$v_{np}(\beta) = e^{i|n|\alpha + ip\beta}, \tag{1}$$

where α is a function of β that will be specified shortly. The representation is a (modified) Fourier transform: the reason for the modulus signs around n will soon be clear.

For $n>0$, we have

$$\begin{aligned} &4v_{np}(\beta)-v_{n+1,p}(\beta)-v_{n-1,p}(\beta)-v_{n,p+1}(\beta)-v_{n,p-1}(\beta) \\ &\quad = e^{in\alpha+ip\beta}[4-e^{-i\alpha}-e^{i\alpha}-e^{-i\beta}-e^{i\beta}] \\ &\quad = 2e^{in\alpha+ip\beta}[2-\cos\alpha-\cos\beta]. \end{aligned}$$

We now require α to be such that

$$\cos\alpha + \cos\beta = 2, \tag{2}$$

so that the above combination of v's vanishes. Similarly, we find zero for this combination if $n<0$. Thus for any integrable $F(\beta)$, $I_{np}=0$, unless $n=0$.

For $0<\beta<2\pi$, there is no real solution of Eq. (2), but there is a complex one, indeed a purely imaginary one, namely

$$\alpha = i\log[2-\cos\beta+\sqrt{3-4\cos\beta+\cos^2\beta}]. \tag{3}$$

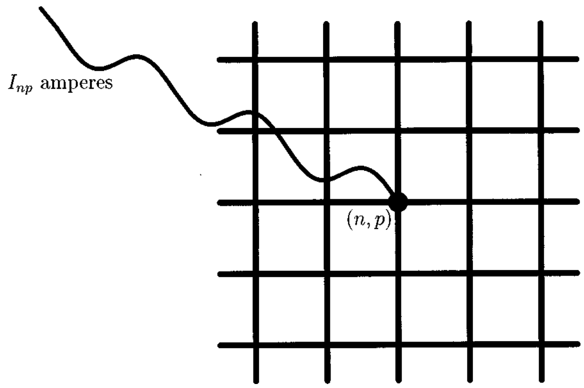

Fig. 1. Infinite square lattice in two dimensions.

Consider now the case $n=0$. We find

$$I_{0p}=\int_0^{2\pi} d\beta F(\beta)e^{ip\beta}[4-2e^{i\alpha}-2\cos\beta]$$

$$=2\int_0^{2\pi} d\beta F(\beta)e^{ip\beta}[\cos\alpha-e^{i\alpha}]$$

$$=-2i\int_0^{2\pi} d\beta F(\beta)\sin\alpha e^{ip\beta}.$$

These currents may be construed as the coefficients of the Fourier series

$$-2iF(\beta)\sin\alpha=\frac{1}{2\pi}\sum_{p=-\infty}^{\infty} I_{0p}e^{-ip\beta},$$

and we may choose the coefficients as we like, thereby specifying $-2iF(\beta)\sin\alpha$. Let $I_{0p}=\delta_{0p}$, i.e., $I_{00}=1$ and $I_{0p}=0$ if $p\neq 0$. This corresponds to the situation in which 1 A enters at (0,0) and leaves at infinity, since no currents leave the lattice at any other finite node. With this choice,

$$F(\beta)=\frac{i}{4\pi\sin\alpha},$$

and so

$$V_{np}=\frac{i}{4\pi}\int_0^{2\pi}\frac{d\beta}{\sin\alpha}e^{i|n|\alpha+ip\beta}. \tag{4}$$

The potential difference between the origin and the node (n,p) is $V_{00}-V_{np}$. This is not quite the resistance between these two points; it is rather one-half of that resistance. To see this, imagine for a moment that we inject 1 A into the node (n,p) instead of (0,0), allowing it to leave at infinity. The new potential at (n,p) will now be what we called V_{00} and the new potential at (0,0) will, by symmetry, be what we called V_{np}. Evidently the new potential difference between the origin and (n,p) is just minus the old potential difference. If we choose to extract 1 A from the node (n,p) instead of injecting it, all the potentials will be simply reversed in sign, so that now the potential difference between the origin and the node (n,p) is again $V_{00}-V_{np}$, just as it was in the original configuration in which the current was injected at (0,0). The final step is to exploit the linearity of Ohm's law and superpose all the currents and potentials appertaining to the configuration in which 1 A enters at (0,0) and that in which 1 A leaves at (n,p). The potential difference in this case is $2[V_{00}-V_{np}]$, and that is therefore equal to R_{np}. Thus

$$R_{np}=\frac{i}{2\pi}\int_0^{2\pi}\frac{d\beta}{\sin\alpha}[1-e^{i|n|\alpha+ip\beta}]. \tag{5}$$

Although the expression (5) does not look symmetric under interchange of n and p, we know from the symmetry of the lattice that it must be so. A formal proof that $R_{np}=R_{pn}$ can be found in Appendix A.

We may transform Eq. (5) into the manifestly real form

$$R_{np}=\frac{1}{\pi}\int_0^{\pi}\frac{d\beta}{\sinh|\alpha|}[1-e^{-|n\alpha|}\cos p\beta]. \tag{6}$$

In Table I we give the results of an evaluation of the expression (6), by means of Mathematica, for all values of n and p between 0 and 5. The program can be found in Appendix B. The method and results of this section may be seen as a streamlining of the technique introduced in the appendix of Ref. 1. This is the formalism that readily lends itself to generalization, as we now proceed to show.

III. THREE DIMENSIONS AND MORE

A cubic infinite lattice of unit resistors in three dimensions can be treated in analogous fashion. There are now three indices, and the current entering the (npq) node is related to the potentials by

Table I. Resistance R_{np} in infinite square lattice.

	p					
n	0	1	2	3	4	5
0	0	$\frac{1}{2}$	$2-\frac{4}{\pi}$	$\frac{17}{2}-\frac{24}{\pi}$	$40-\frac{368}{3\pi}$	$\frac{401}{2}-\frac{1880}{3\pi}$
1	$\frac{1}{2}$	$\frac{2}{\pi}$	$-\frac{1}{2}+\frac{4}{\pi}$	$\frac{46}{3\pi}-4$	$\frac{80}{\pi}-\frac{49}{2}$	$\frac{6646}{15\pi}-140$
2	$2-\frac{4}{\pi}$	$-\frac{1}{2}+\frac{4}{\pi}$	$\frac{8}{3\pi}$	$\frac{1}{2}+\frac{4}{3\pi}$	$6-\frac{236}{15\pi}$	$\frac{97}{2}-\frac{2236}{15\pi}$
3	$\frac{17}{2}-\frac{24}{\pi}$	$\frac{46}{3\pi}-4$	$\frac{1}{2}+\frac{4}{3\pi}$	$\frac{46}{15\pi}$	$\frac{24}{5\pi}-\frac{1}{2}$	$\frac{998}{35\pi}-8$
4	$40-\frac{368}{3\pi}$	$\frac{80}{\pi}-\frac{49}{2}$	$6-\frac{236}{15\pi}$	$\frac{24}{5\pi}-\frac{1}{2}$	$\frac{352}{105\pi}$	$\frac{1}{2}+\frac{40}{21\pi}$
5	$\frac{401}{2}-\frac{1880}{3\pi}$	$\frac{6646}{15\pi}-140$	$\frac{97}{2}-\frac{2236}{15\pi}$	$\frac{998}{35\pi}-8$	$\frac{1}{2}+\frac{40}{21\pi}$	$\frac{1126}{315\pi}$

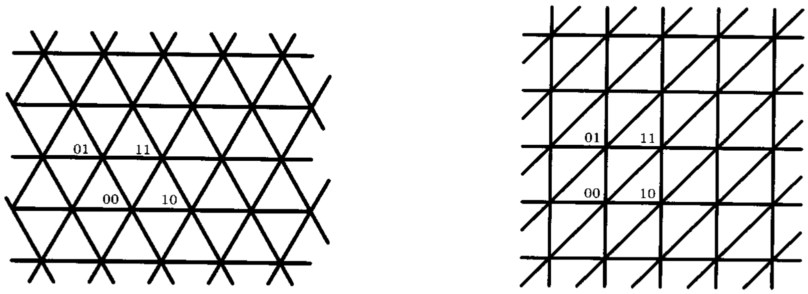

Fig. 2. Triangular lattice.

$$I_{npq} = 6V_{npq} - V_{n+1,p,q} - V_{n-1,p,q} - V_{n,p+1,q} - V_{n,p-1,q} - V_{n,p,q+1} - V_{n,p,q-1}.$$

We set

$$V_{npq} = \int_0^{2\pi} d\beta \int_0^{2\pi} d\gamma F(\beta,\gamma) v_{npq}(\beta,\gamma), \tag{7}$$

where

$$v_{npq}(\beta,\gamma) = e^{i|n|\alpha + ip\beta + iq\gamma},$$

with

$$\cos\alpha + \cos\beta + \cos\gamma = 3, \tag{8}$$

i.e., $\alpha = \cos^{-1}(3 - \cos\beta - \cos\gamma)$.

It is easy to prove that, for $n \neq 0$,

$$I_{npq} = 2\int_0^{2\pi} d\beta \int_0^{2\pi} d\gamma F(\beta,\gamma) e^{i|n|\alpha + ip\beta + iq\gamma} \times [3 - \cos\alpha - \cos\beta - \cos\gamma] = 0,$$

whereas

$$I_{0pq} = -2i \int_0^{2\pi} d\beta \int_0^{2\pi} d\gamma F(\beta,\gamma) \sin\alpha \cos p\beta \cos q\gamma.$$

The inverse of this is the double Fourier series

$$-2iF(\beta,\gamma)\sin\alpha = \frac{1}{4\pi^2} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} I_{0pq} e^{-ip\beta - iq\gamma},$$

and we choose $I_{000} = 1$ and $I_{0pq} = 0$ unless p and q both vanish. This gives

$$F(\beta,\gamma) = \frac{i}{8\pi^2 \sin\alpha},$$

which can be substituted into Eq. (7) to yield the potential V_{npq}. As before, we can compute the resistance corresponding to the situation in which 1 A enters at the origin and leaves at the node (*npq*). It is

$$R_{npq} = \frac{i}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{d\beta d\gamma}{\sin\alpha} [1 - e^{i|n|\alpha + ip\beta + iq\gamma}]. \tag{9}$$

This expression is symmetric under any permutation of the indices, although the symmetries are not manifest (except that under exchange of p and q). A manifestly real form is

$$R_{npq} = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \frac{d\beta d\gamma}{\sinh|\alpha|} [1 - e^{-|n\alpha|} \cos p\beta \cos q\gamma]. \tag{10}$$

This has been integrated numerically (see Appendix B for the program). Some typical results are given below:

$$R_{100} = R_{010} = R_{001} = \tfrac{1}{3},$$

$$R_{110} = R_{101} = R_{011} = 0.39507915,$$

$$R_{111} = 0.41830531, \quad R_{222} = 0.46015929,$$

$$R_{012} = 0.43359881, \quad \text{etc.}$$

The further generalization to four and more dimensions is straightforward. One can simulate a four-dimensional lattice by means of an infinite set of three-dimensional lattices, such that corresponding vertices of the different lattices are linked by one ohm wires. In principle, the same can be done in five or even more dimensions: the fact that we only seem to have three physical spatial dimensions does not limit the dimensionality of physically simulable lattices. In a $(d+1)$-dimensional hypercubic lattice, the resistance between the origin and the $(n, p_1, \ldots, p_d)$ node is

$$R_{np_1\cdots p_d} = \frac{i}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d\beta_1 \cdots d\beta_d}{\sin\alpha} \times [1 - e^{i|n|\alpha + ip_1\beta_1 + \cdots + ip_d\beta_d}], \tag{11}$$

where

$$\cos\alpha + \cos\beta_1 + \cdots + \cos\beta_d = d. \tag{12}$$

As a direction of further generalization, consider the cubic lattice with different resistances in the three directions: say 1 Ω in the *x* direction, $1/\lambda$ Ω in the *y* direction, and $1/\mu$ Ω in the *z* direction. The current entering the (*npq*) node is now

$$I_{npq} = 2(1 + \lambda + \mu)V_{npq} - V_{n+1,p,q} - V_{n-1,p,q} - \lambda V_{n,p+1,q} - \lambda V_{n,p-1,q} - \mu V_{n,p,q+1} - \mu V_{n,p,q-1},$$

and the resistance R_{npq} is still given by Eq. (10), but with

$$\alpha = \cos^{-1}(1 + \lambda + \mu - \lambda\cos\beta - \mu\cos\gamma).$$

With $\lambda = 1 = \mu$ we recover the symmetric cubic lattice, while $\lambda = 1$ and $\mu = 0$ gives the square lattice of the previous section. Evidently $\lambda \neq 1$ and $\mu = 0$ corresponds to a ''rectangular'' lattice, i.e., a square lattice with unequal resistances in the two coordinate directions.

IV. TRIANGULAR LATTICE

We return to two dimensions and consider in this section a triangular lattice. The first question is the choice of the coordinate axes. In Fig. 2 we illustrate our convention: one axis is horizontal, and the other is inclined at 120°, as is indicated by a number of coordinate assignments in the figure. Topologically, the triangular lattice is equivalent to the square lattice with one, but not both diagonals connected, as can also be seen in Fig. 2.

We again seek a representation

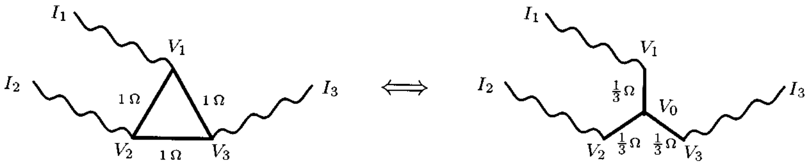

Fig. 3. $\Delta - Y$ transformation.

$$V_{np}=\int_0^{2\pi} d\beta F(\beta) v_{np}(\beta),$$

but we now set

$$v_{np}(\beta)=\exp\left[\frac{i|n-p|\alpha}{2}+\frac{i(n+p)\beta}{2}\right],$$

where β is a real number lying in the interval $(0,2\pi)$, but α will again turn out to be complex. The reason for choosing this form, rather than Eq. (1), is that with our convention the triangular lattice is invariant under a change of sign of $n-p$, but not under a simple change of the sign of n. This can be readily seen from the second of the diagrams in Fig. 2.

We see that the current entering the node (n,p) from outside is related to the potentials by

$$I_{np}=6V_{np}-V_{n+1,p}-V_{n-1,p}-V_{n,p+1}-V_{n,p-1} -V_{n+1,p+1}-V_{n-1,p-1}.$$

For $n-p>0$, we have

$$\begin{aligned}6v_{np}(\beta)&-v_{n+1,p}(\beta)-v_{n-1,p}(\beta)-v_{n,p+1}(\beta)-v_{n,p-1}(\beta)\\ &-v_{n+1,p+1}(\beta)-v_{n-1,p-1}(\beta)\\ &=2v_{np}(\beta)\left[3-\cos\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}-\cos\beta\right],\end{aligned}$$

and this vanishes if

$$2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\beta=3, \tag{13}$$

that is, if α satisfies

$$\alpha=2\cos^{-1}\left(2\sec\frac{\beta}{2}-\cos\frac{\beta}{2}\right), \tag{14}$$

so that under this condition, $I_{np}=0$ if $n>p$. Similarly, the currents vanish if $n<p$; and so we are left to consider the case $n=p$. We find

$$\begin{aligned}I_{nn}&=\int_0^{2\pi} d\beta F(\beta) e^{in\beta}\left[6-2\exp\frac{i(\alpha+\beta)}{2} -2\exp\frac{i(\alpha-\beta)}{2}-2\cos\beta\right]\\ &=2\int_0^{2\pi} d\beta F(\beta) e^{in\beta}\left[\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha-\beta}{2} -\exp\frac{i\alpha}{2}\cos\frac{\beta}{2}\right]\\ &=-4i\int_0^{2\pi} d\beta F(\beta) e^{in\beta}\sin\frac{\alpha}{2}\cos\frac{\beta}{2}.\end{aligned}$$

The currents I_{nn} are the coefficients of the Fourier series

$$-4iF(\beta)\sin\frac{\alpha}{2}\cos\frac{\beta}{2}=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} I_{nn}e^{-in\beta}.$$

The choice $I_{nn}=\delta_{n0}$ yields

$$F(\beta)=\frac{i}{8\pi[\sin(\alpha/2)][\cos(\beta/2)]},$$

and hence

$$V_{np}=\frac{i}{8\pi}\int_0^{2\pi}\frac{d\beta}{[\sin(\alpha/2)][\cos(\beta/2)]}v_{np}(\beta). \tag{15}$$

The resistance between the point (np) and the origin is

$$R_{np}=\frac{i}{4\pi}\int_0^{2\pi}\frac{d\beta}{[\sin(\alpha/2)][\cos(\beta/2)]} \times\left[1-\exp\left(\frac{i|n-p|\alpha}{2}+\frac{i(n+p)\beta}{2}\right)\right]. \tag{16}$$

Setting $y=\beta/2$ and $x=|\alpha|/2=\cosh^{-1}(2\sec y-\cos y)$, we obtain the following real integral:

$$R_{np}=\frac{1}{\pi}\int_0^{\pi/2}\frac{dy}{\sinh x\cos y}[1-e^{-|n-p|x}\cos(n+p)y]. \tag{17}$$

The program for this case is also to be found in Appendix B. It yields

$$R_{10}=R_{01}=R_{11}=\tfrac{1}{3},$$

$$R_{20}=R_{02}=R_{22}=\frac{8}{3}-\frac{4\sqrt{3}}{\pi},$$

$$R_{12}=R_{21}=-\frac{2}{3}+\frac{2\sqrt{3}}{\pi},$$

$$R_{13}=R_{31}=R_{23}=R_{32}=-5+\frac{10\sqrt{3}}{\pi},$$

$$R_{30}=R_{03}=R_{33}=27-\frac{48\sqrt{3}}{\pi}.$$

V. HEXAGONAL LATTICE

The hexagonal lattice may be constructed from the triangular one by an application of the so-called $\Delta-Y$ transformation.[10] The symmetric form that we need is illustrated in Fig. 3.

The triangle, made out of three 1 Ω resistors, is equivalent to the Y form, made out of three $\frac{1}{3}$ Ω resistors, in the sense that the external currents, I_1, I_2, and I_3, and the peripheral potentials, V_1, V_2, and V_3, are the same: thus one circuit may be replaced by the other, so far as the distribution of currents and potentials in the rest of the lattice is concerned. In the Y form, the potential at the middle point is V_0

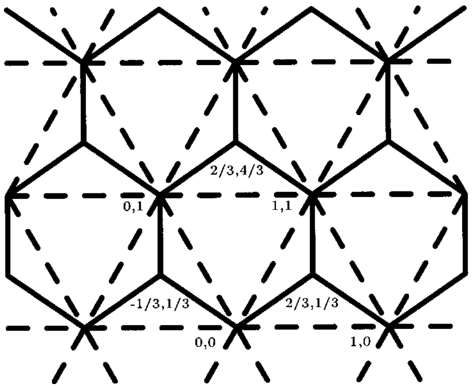

Fig. 4. Hexagonal lattice.

$=\frac{1}{3}[V_1+V_2+V_3]$, on condition that no current enters or leaves the circuit at this point from or to the outside.

In Fig. 4 we see how the triangular lattice of the previous section (dotted lines) may be replaced by a hexagonal lattice (full lines) by a repeated application of the $\Delta-Y$ transformation. Note that the equivalent resistors in the latter case are all $\frac{1}{3}\,\Omega$, so we will need to multiply the final answer for R_{np} by 3 in order to renormalize to the desired case of a hexagonal lattice of 1 Ω resistors. In Fig. 4, we have retained the same coordinates for the nodes that are common to the triangular lattice, and have assigned the relevant fractional n and p values to the new lattice points, the middle points of the Y's.

At the integral coordinate nodes of the hexagonal lattice, the distribution of potentials that corresponds to a current of 1 A entering the node (0,0), and leaving at infinity, is equal to the distribution corresponding to the same situation for the triangular lattice, namely the V_{np} of Eq. (15). The potentials at the fractional coordinate nodes are given by

$$V_{n+2/3,p+1/3}=\tfrac{1}{3}[V_{np}+V_{n+1,p}+V_{n+1,p+1}]. \tag{18}$$

The potential difference between the origin and the node (n,p) is $V_{00}-V_{np}$, and, as before, this is one-half of the resistance between (0,0) and (n,p), but since this applies to a hexagonal lattice of $\frac{1}{3}\,\Omega$ resistors, the final result for a lattice of 1 Ω resistors is three times this value, i.e.,

$$R_{np}=6[V_{00}-V_{np}],$$

where V_{np} is as in Eq. (15). This means that, for integral coordinate nodes, the resistances in the hexagonal lattice are thrice the corresponding resistances in the triangular lattice. For the fractional coordinates, we have

$$\begin{aligned}R_{n+2/3,p+1/3}&=6[V_{00}-V_{n+2/3,p+1/3}]\\&=2[(V_{00}-V_{np})+(V_{00}-V_{n+1,p})\\&\quad+(V_{00}-V_{n+1,p+1})]\\&=\tfrac{1}{3}[R_{np}+R_{n+1,p}+R_{n+1,p+1}].\end{aligned}$$

That is, the resistance from the origin to one of the fractional coordinate nodes is the average of the resistances to the three adjacent integral coordinate nodes. At first sight, one might object that, since the current that enters at (0,0) leaves at $(n+\frac{2}{3},p+\frac{1}{3})$, we have violated the requirement that no current should enter or leave the lattice at the middle point of a Y circuit. However, this objection is unfounded, since, in the original configuration, a current of 1 A enters at (0,0) and leaves at infinity; and in this case no currents enter or leave at any other finite nodes, in particular not at $(n+\frac{2}{3},p+\frac{1}{3})$. This allows us to calculate $V_{00}-V_{n+2/3,p+1/3}$, and that is all we need to complete the calculation of $R_{n+2/3,p+1/3}$.

Typical results are

$$R_{-1/3,1/3}=R_{-1/3,-2/3}=R_{2/3,1/3}=\tfrac{2}{3},$$

$$R_{10}=R_{01}=R_{11}=R_{-1,0}=1,$$

$$R_{2/3,-2/3}=R_{2/3,4/3}=\frac{2\sqrt{3}}{\pi},$$

$$R_{5/3,1/3}=R_{5/3,4/3}=\frac{7}{3}-\frac{2\sqrt{3}}{\pi},$$

$$R_{12}=R_{21}=-2+\frac{6\sqrt{3}}{\pi},$$

$$R_{20}=R_{02}=R_{22}=8-\frac{12\sqrt{3}}{\pi},$$

$$R_{2/3,7/3}=R_{5/3,7/3}=R_{5/3,-2/3}=-3+\frac{8\sqrt{3}}{\pi},$$

$$R_{8/3,5/3}=-\frac{32}{3}+\frac{22\sqrt{3}}{\pi},$$

$$R_{8/3,1/3}=R_{8/3,7/3}=\frac{74}{3}-\frac{42\sqrt{3}}{\pi},$$

$$R_{30}=R_{03}=R_{33}=81-\frac{144\sqrt{3}}{\pi},$$

$$R_{13}=R_{31}=R_{23}=R_{32}=-15+\frac{30\sqrt{3}}{\pi}.$$

APPENDIX A

We shall give a proof that $R_{np}=R_{pn}$ for non-negative n and p: this is sufficient, since from Eq. (6) we see that R_{np} is invariant under the changes $n\to-n$ and/or $p\to-p$. For $n\geqslant 0$ we can write

$$R_{np}=\frac{i}{2\pi}\int_0^{2\pi}\frac{d\beta}{\sin\alpha}[1-e^{i(n\alpha+p\beta)}]. \tag{19}$$

From Eq. (2) we see that

$$\frac{d\beta}{\sin\alpha}=-\frac{d\alpha}{\sin\beta}, \tag{20}$$

and if we can prove that

$$R_{np}=\frac{i}{2\pi}\int_0^{2\pi}\frac{d\alpha}{\sin\beta}[1-e^{i(n\alpha+p\beta)}], \tag{21}$$

we shall have completed our demonstration, since by renaming α as β and β as α, we regain the form of Eq. (19), excepting only that n and p are interchanged.

In view of Eq. (20), it only remains to show that the integration domain $0<\beta<2\pi$ can be transformed into the domain $2\pi>\alpha>0$. To do this, let us change the integration variable from β to

$$\omega=e^{i\beta},$$

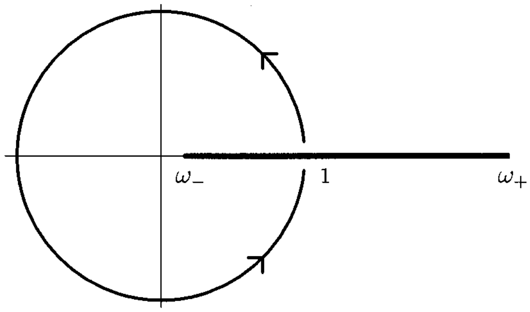

Fig. 5. Original contour in the ω-plane.

in terms of which

$$\sin\alpha = i\,\frac{1-\omega}{2\omega}\sqrt{\omega^2-6\omega+1}. \tag{22}$$

The integral in Eq. (19) is now a contour integral around the unit circle in the ω-plane, from $\omega=1+i\varepsilon$ to $\omega=1-i\varepsilon$, described in the positive sense (see Fig. 5). The square root in Eq. (22) produces a cut, $\omega_- < \omega < \omega_+$, with

$$\omega_\pm = 3 \pm 2\sqrt{2}.$$

Note that $\omega=\omega_-$ corresponds precisely to $\alpha=\pi$.

We now deform the contour of integration from the open unit circle so that it wraps around part of the cut (see Fig. 6). The end points remain where they were, but now the integration goes from unity down to ω_-, along the top of the cut, and then underneath, back up to unity.

On the new contour, α is real, since

$$\cos\alpha = 1 - \frac{(1-\omega)^2}{2\omega},$$

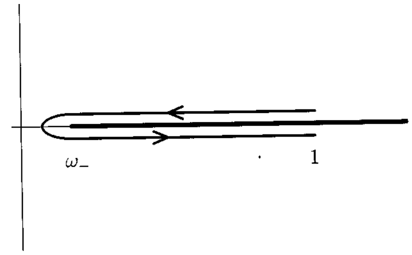

Fig. 6. Deformed contour in the ω-plane.

which goes from 1 at $\omega=1$ to -1 at $\omega=\omega_-$. Moreover, we can cast Eq. (22) into the form

$$\sin\alpha = \mp\frac{1-\omega}{2\omega}\sqrt{(\omega_+-\omega)(\omega-\omega_-)}, \tag{23}$$

the $-$ sign applying above and the $+$ sign below the cut. Thus on the new contour, α begins with the value 0 at $\omega=1-i\epsilon$, progresses to $\alpha=\pi$ at $\omega=\omega_-$, and then continues up to $\alpha=2\pi$ at $\omega=1+i\epsilon$, above the cut, where $\sin\alpha$ is negative. Since we have traversed the contour in the sense *opposed* to the arrow, we have accounted for the minus sign in Eq. (20), and have thereby completed the proof.

APPENDIX B

The resistance in the square lattice, R_{np} of Eq. (6), is programed below as a Mathematica function `Rsqu[n,p]`. The integral is performed symbolically, the answer being exact. Similarly, `Rtri[n,p]` is the resistance for the triangular lattice, as given in Eq. (17). The function `Rcub[n,p,q]` is the corresponding expression for the cubic lattice in three dimensions [see Eq. (9)], but in this case we have been content with numerical integration. The resistances in the hexagonal lattice can be readily obtained from `Rtri[n,p]`.

```
Mathematica Functions for Infinite Lattices
(* Square lattice in 2 Dimemsions *)
alphas[beta_]:=Log[2-Cos[beta]+Sqrt[3+Cos[beta]*(Cos[beta]-4)]];
Rsqu[n_,p_]:=Simplify[(1/Pi)*Integrate[(1-Exp[-Abs[n]*alphas[beta]]*
        Cos[p*beta])/Sinh[alphas[beta]],{beta,0,Pi}]];

(* Triangular lattice in 2 Dimensions *)
alphat[beta_]:=ArcCosh[2/Cos[beta]-Cos[beta]];
Rtri[n_,p_]:=Simplify[1/(Pi)*Integrate[
(1-Exp[-Abs[n-p]*alphat[beta]]*Cos[(n+p)*beta])
/(Cos[beta]*Sinh[alphat[beta]]),{beta,0,Pi/2}]];

(* Cubic lattice in 3 Dimensions *)
alphac[beta_,gamma_]:=ArcCosh[3-Cos[beta]-Cos[gamma]];
Rcub[n_,p_,q_]:=(1/Pi^2)*NIntegrate[NIntegrate[
(1-Exp[-n*alphac[beta,gamma]]*Cos[p*beta]*Cos[q*gamma])
/Sinh[alphac[beta,gamma]],{gamma,0,Pi}],{beta,0,Pi}];
```


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USEFUL KNOWLEDGE

If useful knowledge is, as we agreed provisionally to say, knowledge which is likely, now or in the comparatively near future, to contribute to the material comfort of mankind, so that mere intellectual satisfaction is irrelevant, then the great bulk of higher mathematics is useless. Modern geometry and algebra, the theory of numbers, the theory of aggregates and functions, relativity, quantum mechanics—no one of them stands the test much better than another, and there is no real mathematician whose life can be justified on this ground. If this be the test, then Abel, Riemann, and Poincaré wasted their lives; their contribution to human comfort was negligible, and the world would have been as happy a place without them.

G. H. Hardy, *A Mathematician's Apology* (Cambridge University Press, 1969; reprint of 1940 edition), pp. 135–136.